CHARACTERIZABLE AND V-CHARACTERIZABLE GROUPS

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Abstract

Let $G$ be a finite group. The spectrum $\pi_e(G)$ is the set of all element orders of $G$. A vanishing element of $G$ is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character $\chi$ of $G$. Denote by $\text{Vo}(G)$ the set of the orders of vanishing elements of $G$. For a set $\Omega$ of positive integers, let $h(\Omega)$ ($v(\Omega)$) be the number of isomorphism classes of finite group $G$ such that $\pi_e(G) = \Omega$ ($\text{Vo}(G) = \Omega$, respectively). A group $G$ is called characterizable (V-characterizable) if $h(\pi_e(G)) = 1$ ($v(\text{Vo}(G)) = 1$, respectively). In this note, we discuss the relation between characterizable and V-recognizable. Moreover, by an application of the relation, we prove that the group $M_{22}$ is V-characterizable.

Key words: finite groups, characters, vanishing elements

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1. Introduction. Given a finite group $G$, a vanishing element of $G$ is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character $\chi$ of $G$. We will denote by $\text{Van}(G)$ the set of vanishing elements of $G$. Denote by $\pi(G)$ the set of prime divisors of the orders of $G$ and by $\pi_e(G)$ the set of the element

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orders of $G$. Our aim in this paper is to analyze a particular subset of $\pi_e(G)$, the set $\text{Vo}(G)$ of the orders of elements in $\text{Van}(G)$. We know that $\text{Vo}(G)$ can encode some information about the structure of $G$ (see $[1-3]$).

For a set $\Omega$ of positive integers, let $h(\Omega)$ be the number of isomorphism classes of finite group $G$ such that $\pi_e(G) = \Omega$. A group $G$ is called characterizable if $h(\pi_e(G)) = 1$. We define a similar concept for the set of orders of vanishing elements.

**Definition 1.1.** For a set $\Omega$ of positive integers, let $v(\Omega)$ be the number of isomorphism classes of finite group $G$ such that $\text{Vo}(G) = \Omega$. A non-abelian group $G$ is called V-characterizable if $v(\text{Vo}(G)) = 1$.

Given a finite set of positive integers $X$, the prime graph $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes $p$ such that there exists an element of $X$ divisible by $p$, and two distinct vertices $p$ and $q$ are adjacent if and only if there exists an element of $X$ divisible by $p$ and $q$. For a finite group $G$, the graph $\Pi(\pi_e(G))$, which we denote by $GK(G)$, is also known as the Gruenberg–Kegel graph of $G$. We focus our attention on a particular subgraph of $GK(G)$. The prime graph $\Pi(\text{Vo}(G))$ which in this paper we denote by $\Gamma(G)$, is called the vanishing prime graph of $G$. The vanishing prime graph was introduced in $[4,5]$.

The relation between $h(\pi_e(G))$ and $v(\text{Vo}(G))$, we give in the following result.

**Theorem 1.2.** Let $G$ be a finite group and let $H$ be a simple group of Lie type or a sporadic simple group with $\text{Vo}(G) = \text{Vo}(H)$. Assume that $\Gamma(H)$ is disconnected, and that $N$ is a normal subgroup of $G$ and that $G/N \cong H$. If $N > 1$, then $G$ has a normal series

$$1 \leq V < N \leq G,$$

where $N/V$ is an elementary abelian $p$-group, for some prime $p$ and

$$\pi_e(G/V) = \pi_e(H).$$

In particular, if $H$ is characterizable, then $H$ is V-characterizable.

As an application of Theorem 1.2, we obtain the following result:

**Theorem 1.3.** Let $G$ be a finite group. Then $G \cong M_{22}$ if and only if $\text{Vo}(G) = \text{Vo}(M_{22})$.

**Remark 1.4.** There exists non-abelian simple groups which are not V-recognizable. For the simple linear group $L_2(9)$, let $G = NA$, where $A \cong \text{SL}_2(4)$ and $N$ is an elementary abelian 2-group and a direct sum of natural $\text{SL}_2(4)$-modules, we easily conclude that $\text{Vo}(G) = \text{Vo}(L_2(9))$. In particular, $v(\text{Vo}(L_2(9))) = \infty$. Also, for the simple linear group $L_3(5)$, $\text{Vo}(L_3(5)) = \text{Vo}(\text{Aut}(L_3(5)))$, but $L_3(5) \not\cong \text{Aut}(L_3(5))$ because $|L_3(5)| \neq |\text{Aut}(L_3(5))|$.

In fact, we also know that both $L_2(9)$ and $L_3(5)$ are not recognizable. Naturally, we may ask the following question:
\textbf{Question 1.5.} Let $G$ be a finite group and let $H$ be a finite non-abelian simple group with $\text{Vo}(G) = \text{Vo}(H)$. If $H$ is recognizable, then $G \cong H$? 

All further unexplained notation is standard, readers may refer to [6] and [7].

\textbf{2. Preliminary results.} The following lemma provides some properties of the vanishing prime graph of a finite group and its relationship with the Gruenberg–Kegel graph. In what follows, we shall denote by $V(G)$ the vertex set of a graph $G$, and by $n(G)$ the number of connected components of $G$.

\textbf{Lemma 2.1} ([4,5]). Let $G$ be a finite group. Then the following hold:

1. If $G$ is solvable, then $\Gamma(G)$ has at most two connected components.
2. If $G$ is non-solvable and $\Gamma(G)$ is disconnected, then $G$ has a unique non-abelian composition factor $S$, and $n(\Gamma(G)) \leq n(GK(S))$ unless $G$ is isomorphic to $A_7$.

\textbf{Lemma 2.2} ([7], Proposition 2.1). Let $G$ be a non-abelian simple group and $p$ a prime number. If $G$ is of Lie type, or if $p \geq 5$, then there exists $\chi \in \text{Irr}(G)$ of $p$-defect zero.

In the following lemma, we collect some basic remarks relating the vanishing elements of a group $G$ and the vanishing elements of the quotients of $G$. We shall freely use these results.

\textbf{Lemma 2.3} ([4,8]). Let $N$ be a normal subgroup of $G$.

1. Any character of $G/N$ can be viewed, by inflation, as a character of $G$. In particular, if $xN \in \text{Van}(G/N)$, then $xN \subseteq \text{Van}(G)$.
2. If $p \in \pi(N)$ and $N$ has an irreducible character of $p$-defect zero, then every element of $N$ of order divisible by $p$ is a vanishing element of $G$.
3. If $m \in \text{Vo}(G/N)$, then there exists an integer $n$ such that $mn \in \text{Vo}(G)$.

\textbf{Remark 2.4.} Let $G$ be a simple group of Lie type. By Lemma 2.2, $G$ has characters of $p$-defect zero for every prime $p$ and hence by [7], Theorem 8.17 every non-identity element of $G$ is a vanishing element. Hence $\text{Vo}(G) = \pi_e(G) - \{1\}$.

\textbf{3. The relation between characterizable and V-recognizable.} 

\textbf{Lemma 3.1} ([4], Proposition 4.2). Let $G$ be a group. Assume that $V(\Gamma(G)) \neq \pi(G)$. Then $\Gamma(G)$ is connected. Moreover, if $G$ is non-solvable, then $G$ has an unique non-abelian composition factor $S$ and $S \cong A_5$.

Next, we first give a case which forces that $\text{Vo}(G)$ and $\pi_e(G)$ are almost the same.

\textbf{Lemma 3.2} ([8], Lemma 3.2). Let $G$ be a finite group, and let $N$ be a normal subgroup of $G$. Assume that every non-identity element of $G/N$ is a vanishing element of $G/N$. If $N$ is an elementary abelian $p$-group for some prime $p$, then

$$\pi_e(G) = \text{Vo}(G) \cup \{1,p\}.$$ 

Moreover, if $\Gamma(G)$ is disconnected, then $p \in V(\Gamma(G))$.

In the following proof of Theorem 1.2, we will use the fact: let $G$ and $H$ be two groups, if $\text{Vo}(G) = \pi_e(H) - \{1\}$, then, for any $m \in \text{Vo}(G)$ and any non-identity factor $n$ of $m$, we have that $n \in \text{Vo}(G)$.
Proof of Theorem 1.2. Recall that $\text{Vo}(G) = \text{Vo}(H)$ and that $\Gamma(H)$ is disconnected, then it follows by the hypothesis and Lemma 2.1, that $N$ is the solvable radical of $G$.

Assume that $N > 1$. Let $1 \leq V < N$ such that $N/V$ is a chief factor of $G$. Since $N$ is solvable, $N/V$ is an elementary abelian $p$-group, for some prime $p$. Now, we consider the group $\tilde{G} := G/V$.

First, assume that $H$ satisfies the condition (*): every non-identity element of $H$ is a vanishing element of $H$. Then applying Lemma 3.2 in the group $\tilde{G}$, we get

$$\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, p\}.$$ 

Recall that every non-identity element of $G/N$ is a vanishing element of $G/N$, thus $\text{Vo}(G/N) = \pi_e(G/N) - \{1\}$. Moreover, $\text{Vo}(G) = \text{Vo}(G/N)$, hence we conclude

$$\text{Vo}(G) = \pi_e(G/N) - \{1\} = \text{Vo}(G/N).$$

By the first equation above, we have that all of non-identity factors of each element of $\text{Vo}(G)$ are also included in $\text{Vo}(\tilde{G})$. Hence we obtain

$$\text{Vo}(\tilde{G}) \subseteq \text{Vo}(G) = \text{Vo}(G/N) = \text{Vo}(\tilde{G}/\tilde{N}).$$

Therefore, we conclude

$$\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, p\} \subseteq \text{Vo}(\tilde{G}/\tilde{N}) \cup \{1, p\} \subseteq \pi_e(\tilde{G}).$$

So, we have

$$\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}/\tilde{N}) \cup \{1, p\}.$$ 

Recall that $\text{Vo}(G/N) = \pi_e(G/N) - \{1\}$ and that $\text{Vo}(G/N) = \text{Vo}(\tilde{G}/\tilde{N})$, we obtain

$$\pi_e(\tilde{G}) = \pi_e(G/N) \cup \{p\}.$$ 

Since $\Gamma(G)$ is disconnected, it follows by Lemma 3.1 that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G))$. Notice that all of non-identity factors of each element of $\text{Vo}(G)$ are included in $\text{Vo}(G)$ and that $\text{Vo}(G) = \pi_e(G/N) - \{1\}$. Thus we have that $p \in \pi_e(G/N)$. Hence we get

$$\pi_e(\tilde{G}) = \pi_e(G/N) = \pi_e(H).$$

Let $H$ be a simple group of Lie type. Then by Remark 2.4, $H$ satisfies the condition (*). Let $H$ be a sporadic simple group. A direct inspection of [6] shows that except for $M_{22}$ and $M_{24}$, $H$ satisfies the condition (*).

Now, suppose that $H \cong M_{22}$. Notice that any element in $\text{Vo}(\tilde{G})$ is a factor of some element in $\text{Vo}(G)$ and that $\text{Vo}(G) = \text{Vo}(M_{22})$. Hence get that

$$\text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{22}) \cup \{2\} = \{3, 4, 5, 6, 7, 8, 11\} \cup \{2\}.$$
Similarly, any element in $\text{Vo}(\tilde{G}/\tilde{N})$ is a factor of some element in $\text{Vo}(\tilde{G})$ and $\tilde{G}/\tilde{N} \cong M_{22}$. Then we conclude that

\begin{equation}
\{5, 6, 7, 8, 11\} \subseteq \text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{22}) \cup \{2\} = \pi_e(M_{22}) - \{1\}.
\end{equation}

Note that $\pi_e(\tilde{G}) = \pi_e(\tilde{G}/\tilde{N}) \cup \{1, p\}$. For any element $x$ in $\tilde{G}/\tilde{N}$, if $x$ is a non-vanishing element of $\tilde{G}$, then $o(xN) = 2$, and so $o(x) = 2$, or $2p$. Therefore, we obtain that

\begin{equation}
\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, 2, p, 2p\}.
\end{equation}

Since $\Gamma(G)$ is disconnected, it follows by Lemma 3.1 that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G)) = \{2, 3, 5, 7, 11\}$.

Assume that $p \in \{5, 7, 11\}$. Let $S$ be a Sylow 3-subgroup of $\tilde{G}$. Then by (1) and (2), the group $\tilde{N}S$ is a Frobenius group with kernel $\tilde{N}$ and a complement $S$. Then $S$ is cyclic. However, by [9], we obtain a contradiction, which implies that $p \in \{2, 3\}$. Recall that $\{5, 6, 7, 8, 11\} \subseteq \text{Vo}(\tilde{G}) \subseteq \{2, 3, 4, 5, 6, 7, 8, 11\}$, hence we conclude that

\begin{equation}
\pi_e(\tilde{G}) = \{1, 2, 3, 4, 5, 6, 7, 8, 11\} = \pi_e(M_{22}).
\end{equation}

Next, suppose that $H \cong M_{24}$. Notice that any element in $\text{Vo}(\tilde{G})$ is a factor of some element in $\text{Vo}(G)$ and that $\text{Vo}(\tilde{G}) = \text{Vo}(M_{24}) = \pi_e(M_{24}) - \{1\}$. Hence we get that

\begin{equation}
\text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{24}).
\end{equation}

Similarly, any element in $\text{Vo}(\tilde{G}/\tilde{N})$ is a factor of some element in $\text{Vo}(\tilde{G})$ and $\tilde{G}/\tilde{N} \cong M_{24}$. Then we conclude that

\begin{equation}
\{8, 10, 11, 12, 14, 15, 21, 23\} \subseteq \text{Vo}(\tilde{G}) \subseteq \text{Vo}(M_{24}) = \pi_e(M_{24}) - \{1\}.
\end{equation}

Note that $\pi_e(\tilde{G}) = \pi_e(\tilde{G}/\tilde{N}) \cup \{1, p\}$. For any element $x$ in $\tilde{G}/\tilde{N}$, if $x$ is a non-vanishing element, then $o(x) = 2$, or $2p$. Therefore, we obtain that

\begin{equation}
\pi_e(\tilde{G}) = \text{Vo}(\tilde{G}) \cup \{1, 2, p, 2p\}.
\end{equation}

Since $\Gamma(G)$ is disconnected, it follows by Lemma 3.1 that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G)) = \{2, 3, 5, 11, 12, 23\}$.

Assume that $p \in \{11, 23\}$. Let $S$ be a Sylow 3-subgroup of $\tilde{G}$. Then by (3) and (4), the group $\tilde{N}S$ is a Frobenius group with kernel $\tilde{N}$ and a complement $S$. Then $S$ is cyclic. Then by [9], we also obtain a contradiction, which implies that $p \in \{2, 3, 5\}$. Recall that $\{8, 10, 11, 12, 14, 15, 21, 23\} \subseteq \text{Vo}(\tilde{G})$, hence we conclude that

\begin{equation}
\pi_e(\tilde{G}) = \pi_e(M_{24}).
\end{equation}

The proof is completed. \qed

4. An application of Theorem 1.2. In this section, we are ready to prove Theorem 1.3. Denote by $s(\mathcal{H})$ the number of connected components of a graph $\mathcal{H}$. It follows from the classification of simple groups with disconnected vanishing prime graph that $s(\Gamma(H)) \leq 6$ for every finite group $G$ (see [9]). First, using [9], we collect the sporadic simple groups with $s(\Gamma(H)) \geq 4$ in Table 1:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\text{Vo}(H)$</th>
<th>$V(\Gamma(H))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O'N$</td>
<td>$2 - 8, 10 - 12, 14 - 16$, $19, 20, 28, 31$</td>
<td>${2, 3, 5, 7}$, ${11}, {19}, {31}$</td>
</tr>
<tr>
<td>$Ly$</td>
<td>$2 - 12, 14, 15, 18$, $20 - 22, 24, 25, 28, 30$, $31, 33, 37, 40, 42, 67$</td>
<td>${2, 3, 5, 7, 11}$, ${31}, {37}, {67}$</td>
</tr>
<tr>
<td>$F'_{24}$</td>
<td>$2 - 18, 20 - 24, 26 - 30, 33$, $35, 36, 39, 42, 45, 60$</td>
<td>${2, 3, 5, 7, 11}$, ${29}$</td>
</tr>
<tr>
<td>$M$</td>
<td>$2 - 36, 38 - 42, 44 - 48$, $50 - 52, 54 - 57, 59, 60$, $62, 66, 68 - 71, 78, 84, 87$, $88, 92 - 95, 104, 105, 110, 119$</td>
<td>${2, 3, 5, 7, 11}$, ${17}, {23}, {29}$</td>
</tr>
<tr>
<td>$J_1$</td>
<td>$2 - 8, 10 - 12, 14 - 16$, $20 - 24, 28 - 31, 33$, $35, 37, 40, 42 - 44, 66$</td>
<td>${2, 3, 5, 7, 11}$, ${23}, {29}$</td>
</tr>
<tr>
<td>$M_{22}$</td>
<td>$3 - 8, 11$</td>
<td>${2, 3}, {5}, {7}, {11}$</td>
</tr>
<tr>
<td>$J_1$</td>
<td>$2 - 3, 5 - 7, 10, 11, 15, 19$</td>
<td>${2, 3, 5}, {7}, {11}, {19}$</td>
</tr>
</tbody>
</table>

The following result, which appears as in [9], Proposition 3.2, will turn out to be useful in the proof of Theorem 1.3.

**Lemma 4.1.** Let $N$ be normal in $G$. Suppose that $G$ has a normal series

$$1 \leq N < M \leq G,$$

where $M/N \cong \text{PSL}_3(4)$. If only 2 and 3 are connected in $\Gamma(G)$, then $N$ is a 3-group.

**Proof of Theorem 1.3.** Assume that $\text{Vo}(G) = \text{Vo}(M_{22})$. Then by part (1) of Lemma 2.1, $G$ is non-solvable. Let $N$ be the solvable radical of $G$. Then by part (2) of Lemma 2.1, $G$ has a normal series

$$1 \leq N < M \leq G,$$

where $G/M$ is a solvable group, and $M/N$ is a non-cyclic simple group. Now we consider the group $\overline{G} := G/N$. As $N$ is the solvable radical of $G$, $\overline{G} \leq \text{Aut}(M)$ and $G/M \leq \text{Out}(M)$. Recall that $n(\Gamma(G)) = 4$; thus by part (2) of Lemma 2.1, $n(\Gamma(M)) \geq 4$. 

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We show that $\overline{M}$ is a sporadic simple group. Let $\overline{M} \cong A_n$ for some $n \geq 12$. For odd $n$, set

$$a = (1, \ldots, n - 9)(n - 8, n - 7, n - 6, n - 5)(n - 4, n - 3, n - 2, n - 1, n).$$

For even $n$, set

$$a = (1, \ldots, n - 8)(n - 7, n - 6)(n - 5, n - 4, n - 3, n - 2, n - 1).$$

By Lemma 2.2, there exists $\chi \in \text{Irr}(\overline{M})$ such that $\chi$ is of $5$-defect zero. Then, by Lemma 2.3(2), $a \in \text{Van}(G)$. Observe that there exists an element $g \in G$ such that $40 \mid o(g)$, a contradiction. Hence the hypothesis yields that $n < 12$. Then by [6], $n(GK(\overline{M})) \leq 3$, and so we obtain a contradiction (note that $n(GK(\overline{M})) \geq 4$).

By the classification theorem of the finite simple groups, we can now suppose that $\overline{M}$ is a simple group of Lie type. Then by Lemma 2.2, for any prime divisor $p$ of $|\overline{M}|$, there exists $\chi_p \in \text{Irr}(\overline{M})$ such that $\chi_p$ is of $p$-defect zero, and so every element of $\overline{M}$ of order divisible by $p$ is a vanishing element of $G$. Hence every non-identity element of $\overline{M}$ is a vanishing element of $G$, and thus $\overline{M} \setminus N \subseteq \text{Van}(G)$. Therefore get that

$$\pi_e(\overline{M}) - \{1\} = \text{Vo}(\overline{M}) \subseteq \text{Vo}(G) \subseteq \text{Vo}(G) \cup \{2\} = \{2, 3, 4, 5, 6, 7, 8, 11\}.$$

Then $\max\{\pi_e(\overline{M})\} = 11$. So by [10], Table 1, $\overline{M} \cong \text{PSL}_3(4)$ (note that $s(\Pi(\overline{M})) \geq 4$). It follows by the hypothesis and Lemma 4.2 that $N$ is a $3$-group. Recall that $11 \in \pi(G)$, Thus by [6], we obtain a contradiction, which implies that $\overline{M}$ is a sporadic simple group. Recall that $s(\Pi(\overline{M})) \geq 4$, thus $\overline{M}$ is one of the groups in Table 1.

Assume that $\overline{M}$ is not isomorphic to $M_{22}$. Then by [6], for any prime divisor $p$ of $|\overline{M}|$, there exists $\chi_p \in \text{Irr}(\overline{M})$ such that $\chi_p$ is of $p$-defect zero, and so every element of $\overline{M}$ of order divisible by $p$ is a vanishing element of $G$. Hence every non-identity element of $\overline{M}$ is a vanishing element of $G$, and thus any non-identity element of $\overline{M}$ is a factor of some element in $\text{Vo}(G)$, then we obtain a contradiction, which implies that $\overline{M}$ is isomorphic to $M_{22}$.

Since $\overline{M} \cong M_{22}$, by [6], we conclude that $\overline{M} = M$. By [11], the group $M_{22}$ is recognizable, and then by Theorem 1.2, $G \cong M_{22}$, the proof is completed.

REFERENCES


